Часть 1

Олимпиада: **Математика, 11 класс (1 часть)**

Шифр: **21104501**

ID профиля: **101274**

Вариант 21

Чистовик

№1) $S = a_1 + \ldots + a_7$ $\quad d > 0$ $\quad a_{17} \cdot a_8 > S + 27;$ $\quad a_{11} \cdot a_{14} < S + 60$

$\frac{2a_1 + 6d}{2} \cdot 7 = S$ – ф-ла суммы арифм. прогрессии

$7a_1 + 21d = S$

$a_8 = a_{11} - 3d$
$a_{17} = a_{11} + 6d$
$a_{14} = a_{11} + 3d$

$\begin{cases} (a_{11} - 3d)(a_{11} + 6d) > S + 27 \\ a_{11}(a_{11} + 3d) < S + 60 \end{cases}$

$\begin{cases} a_{11}^2 + 3a_{11}d - 18d^2 > S + 27 & (1) \\ a_{11}^2 + 3a_{11}d < S + 60 & (2) \end{cases}$ усилим (1) $((2) \Rightarrow (1))$

$\exists\, a_{11}^2 + 3a_{11}d = u$ $\quad u < S + 60 \Rightarrow$

$S + 60 > u$

$S + 60 - 18d^2 > u - 18d^2$

$S + 60 - 18d^2 > S + 27$

$33 > 18d^2$

$\underset{2}{\overset{\wedge}{\frac{11}{6}}} > d^2 \Rightarrow d = 1$

т.к. $d > 0$ и $d \in \mathbb{Z}$

т.к. $a_1, a_2 \ldots \in \mathbb{Z}$

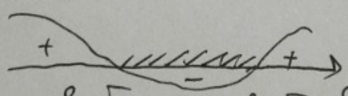
$\begin{cases} (a_1 + 7d)(a_1 + 16d) > S + 27 \\ (a_1 + 10d)(a_1 + 13d) < S + 60 \end{cases}$

$S = 21 + 7a_1$

$\begin{cases} a_1^2 + 16a_1d + 7 \cdot 16d^2 > \cancel{7a}\ 48 \\ a_1^2 + 16a_1d + 130d^2 < 81 \end{cases}$

$\begin{cases} (a_1^2 + 16a_1 + 64) > 0 \\ (a_1^2 + 16a_1 + 49) < 0 \end{cases}$ $\quad (a_1 + 8)^2 > 0 \Rightarrow a_1 \neq -8 \quad (*)$

$D = 256 - 49 \cdot 4 = 60$ $\quad a_1 = \frac{-16 \pm 2\sqrt{15}}{2}$ $\quad \left[\begin{array}{l} a_1 = -8 + \sqrt{15} \\ a_1 = -8 - \sqrt{15} \end{array}\right.$



$-8 - \sqrt{15}$ $\quad -8 + \sqrt{15}$ $\quad a_1$

$\vee$ $\quad \wedge$

-12 $\quad -4$

$a_1 \in \mathbb{Z} \Rightarrow -11; -10; -9; \underline{-8}; -7; -6; -5$

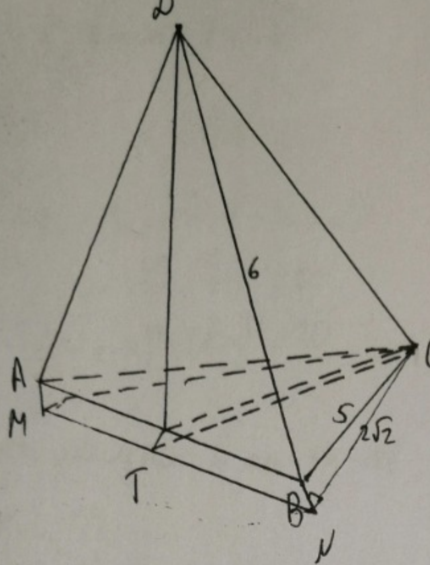
$(*)$ - неподх.

Ответ: $-11, -10, -9; -7; -6; -5$

1

Чистовик

№2)



1) Проведём плоскость CNM так, что $(CMN) \perp CD$

$\Rightarrow BN \parallel AB \parallel CD \Rightarrow AMNB$ пр-к

$\Rightarrow MN = 4 = AB$

2) $\Delta BCN = \Delta AMC$ ($AC = BC$; $BN = AM$; $\angle CMA = \angle CNB$)

$\Rightarrow CN = MC = x$

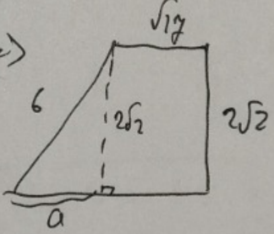
$$R = \frac{x \cdot x \cdot 4}{4 \cdot \frac{1}{2} \cdot 4\sqrt{x^2-4}} = \frac{x^2}{2\sqrt{x^2-4}}$$

$$R' = \frac{2x(2\sqrt{x^2-4}) - x^2 \cdot \frac{2x}{\sqrt{x^2-4}}}{4(x^2-4)} = 0$$

$4x(x^2-4) = 2x^3 \quad 2x^3 = 16x \quad x^2 = 8$

$x = 2\sqrt{2}$

$CN \perp BN \Rightarrow DC \perp CN \Rightarrow$



$BN = \sqrt{17}$ по th. Пифагора

$\sqrt{6^2-8} = a =$

$= \sqrt{36-8} = \sqrt{28} = 2\sqrt{7}$

$CD = \sqrt{17} + 2\sqrt{7}$

Ответ: $\sqrt{17} + 2\sqrt{7}$

Чистовик

№3)

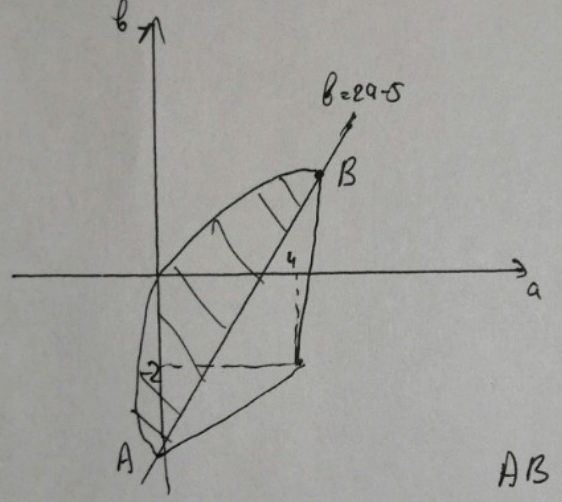
$$\begin{cases} (x-a)^2+(y-b)^2 \le 20 \\ a^2+b^2 \le \min(8a-4b;20) \end{cases}$$

I) $8a-4b<20$ $\qquad b<2a-5$

$$a^2-8a+16+b^2+4b+4\le 20$$

$$\begin{cases} (a-4)^2+(b+2)^2\le 20 \\ (a-x)^2+(b-y)^2\le 20 \end{cases}$$

Построим рисунок в пл-ти a,b



$\measuredangle AB$ $\quad (a-4)^2+(2a-3)^2\le 20$

$a^2+16-8a+4a^2+9-12a\le 20$

$5a^2-20a+5\le 0$

$a^2-4a-1\le 0$

$a=2\pm\sqrt5$

$b=-1\pm2\sqrt5$

$AB=\sqrt{16\cdot5+4\cdot5}=10$

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Черновик

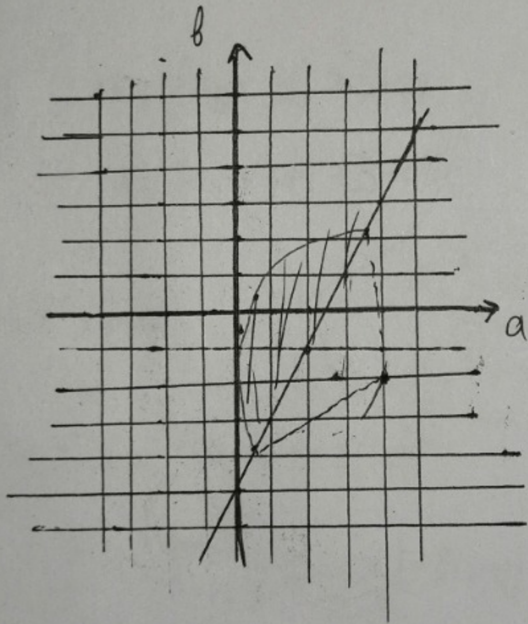
№3)

$$\begin{cases} (x-a)^2+(y-b)^2 \le 20 \\ a^2+b^2 \le \min(8a-4b, 20) \end{cases}$$

1) $\exists \; 8a-4b < 20$

$2a-b<5 \quad \underline{b>2a-5}$

2) $20 < 8a-4b \quad b<2a-5$



$a^2+b^2-8a+4b \le 0$

$a^2-8a+16+b^2+4b+4 \le 20$

$(a-4)^2+(b+2)^2 \le 20$

$(a-4)^2+(2a-3)^2 \le 20$

$a^2-8a+16+4a^2+9-12a \le 20$

$5a^2-20a-5=0$

$a^2-4a-1 \le 0$

$D=16+4=20$

$a=2\pm\sqrt{5}$

$\left[\begin{array}{l} b=2\sqrt{5}-1 \\ b=-2\sqrt{5}-1 \end{array}\right.$

$\Rightarrow$ хорда $l_1 = \sqrt{20+16\cdot5} = \sqrt{20\cdot5}=10$

$\Rightarrow \; 100 = 20+20-2\cdot20\cdot\cos\alpha$

$40\cos\alpha = -60$

$\cos\alpha = -\frac{3}{2}$

$4\sqrt{5}<10$

$2\sqrt{5}<5$

$20<25$

Черновик

№1)

$S = \frac{a_1 + a_7}{2} \cdot 7$

$a_8 \cdot a_{17} > S + 27$

$a_{11} \cdot a_{14} < S + 60$

$\frac{2a_1 + 6d}{2} \cdot 7 = S$

$a_1 + 3d = \frac{S}{7}$

$a_1 = \frac{S}{7} - 3d$

$d = \frac{\frac{S}{7} - a_1}{3}$

$7a_1 + 21d = S$

$6a_1 +$

$(a_1 + 7d)(a_1 + 16d) > S + 27$

$\left(\frac{S}{7} + 4d\right)\left(\frac{S}{7} + 13d\right) > S + 27$

$\frac{S^2}{49} + \frac{45d}{7} + \frac{135d}{7} + 52d^2 > S + 27$

$a_8(a_8 + 9d) > S + 27$

$a_{11}(a_{11} + 3d) \cancel{<} S + 60$

$(a_{11} - 3d)(a_{11} + 6d) > S + 27$

$a_{11}(a_{11} + 3d) \cancel{<} S + 60$

$\underline{a_{11}^2 - 3a_{11}d + 6a_{11}d} - 18d^2 > S + 27$

$\underline{a_{11}^2 + 3a_{11}d} < S + 60$

$S + 60 - 18d^2 > S + 27$

$33 > 18d^2$

$11 > 6d^2$

$\frac{11}{6} > d^2 \quad \Rightarrow d = 1$

$a_1 = \frac{S}{7} - 3d$

$a_1^2 + \cancel{23a_1d + 7 \cdot 16d^2} > S + 27$

a_1

$(3a_1 + 21d)(3a_1 + 48d) > 9(S + 27)$

$(S - 4a_1)\left(\frac{16}{7}S - 13a_1\right) > 9(S + 27)$

$(S - 4a_1)(16S - 13a_1) > 63(S + 27)$

$16S^2 - 77a_1S + 52a_1^2 > 63S + 63 \cdot 27$

$52a_1^2 - 77Sa_1 + 16S^2 - 63S - 63 \cdot 27 > 0$

$D = 77^2 \cdot S^2 - 52 \cdot 4 \cdot 16S^2 + 52 \cdot 4 \cdot 63S + 63 \cdot 27 \cdot 52 \cdot 4$

$$\begin{array}{r} 49 \\ \times\ 4 \\ \hline 196 \end{array} \qquad \begin{array}{r} 256 \\ -196 \\ \hline 60 \end{array}$$

$a_4 = \frac{10}{21}S - \frac{7}{3}a_1$

$(a_1 + 7)(a_1 + 16) > 7a_1 + 48$

$a_1^2 + 23d_1 + 7 \cdot 16 > 7a_1 + 48$

$a_1^2 + 16a_1 + 64 > 0$

$(a_1 + 8)^2 > 0 \quad \Rightarrow a_1 \neq -8$

$(a_1 + 10)(a_1 + 13) < 7a_1 + 81$

$a_1^2 + 23a_1 + 130 - 7a_1 - 81 < 0$

$a_1^2 + 16a_1 + 49 < 0$

$D = 256 - 49 \cdot 4 =$

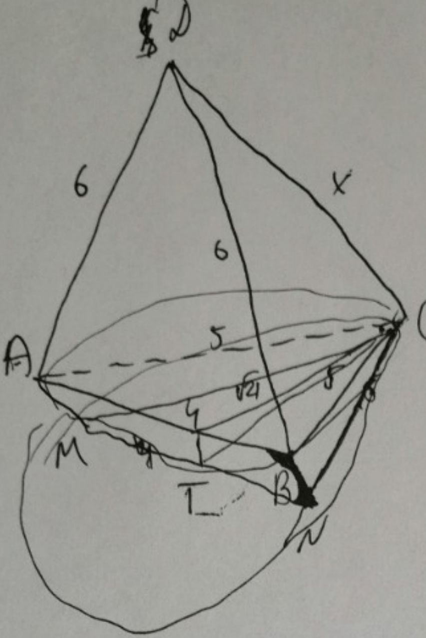
$= 256 - 196 = 60$

$a_1 = \frac{-18 \pm 2\sqrt{15}}{2}$

$-9 - \sqrt{15} \quad -9 + \sqrt{15}$

Черновик

№2)



$\sqrt{36-4} = \sqrt{32} = 4\sqrt{2}$

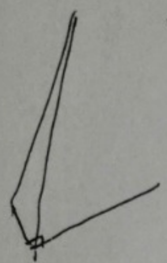
$x^2 = 32+21-2\sqrt{32}\cdot\sqrt{21}\cdot\cos\alpha$

$x^2 = 53 - 2\sqrt{32\cdot 21}\cdot\cos\alpha$

$x = \sqrt{53-8\sqrt{42}\cos\alpha}$

$x' = \frac{1}{2\sqrt{---}}\cdot(\cancel{...} + 8\sqrt{42}\sin\alpha)$

$\Rightarrow \alpha = 90^\circ$



$\Rightarrow \underline{CD = \sqrt{53}}$

$AB = 4$ $\cancel{AC = AC = x}$

$MC = CN = x$

$R = \frac{x\cdot x\cdot 4}{4\cdot\sqrt{x^2-4}\cdot 4\cdot\frac{1}{2}} = \frac{x^2}{2\sqrt{x^2-4}}$

$R' = \frac{2x(2\sqrt{x^2-4}) - x^2\cdot 2\frac{1}{2\sqrt{x^2-4}}\cdot(2x)}{4(x^2-4)} = 0$

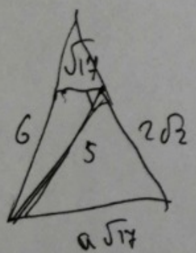
$CT = \sqrt{8-4} = 2$

$4x\sqrt{x^2-4} = \frac{2x^3}{\sqrt{x^2-4}}$

$4x(x^2-4) = 2x^3$

$2x^3 = 16x$

$x^2 = 8$. $x = 2\sqrt{2}$



$$\begin{array}{r} 25 \\ -8 \\ \hline 17 \end{array}$$

21104501 (U101274 M1296831)

Часть 2

Олимпиада: **Математика, 11 класс (2 часть)**

Шифр: **21104501**

ID профиля: **101274**

Вариант 21

Чистовик

№4) $\begin{cases} НОД(a;b;c) = 35 \\ НОК(a;b;c) = 5^{18} \cdot 7^{16} \end{cases}$ $\Rightarrow$ $\begin{matrix} a \vdots 35 \\ b \vdots 35 \\ c \vdots 35 \end{matrix}$ $\Rightarrow$ в разложении на простые множители в каждом есть 5 и 7 хотя бы в 1 ст.

$\exists$ $a = 5^{k_1} \cdot 7^{k_2}$
$b = 5^{l_1} \cdot 7^{l_2}$
$c = 5^{m_1} \cdot 7^{m_2}$

$\Rightarrow \left[\begin{matrix} k_1 = 18 \\ l_1 = 18 \\ m_1 = 18 \end{matrix} \right.$ $\left[\begin{matrix} k_2 = 16 \\ l_2 = 16 \\ m_2 = 16 \end{matrix} \right.$

т.к. НОД $= 35 \Rightarrow$ $\left[\begin{matrix} k_2 = 1 \\ l_2 = 1 \\ m_2 = 1 \end{matrix} \right.$ $\left[\begin{matrix} k_1 = 1 \\ l_1 = 1 \\ m_1 = 1 \end{matrix} \right.$

$k_1 \quad l_1 \quad m_1$

$\underset{3 вар}{18}$ $\underset{2 вар}{1}$ 6 вариантов

для ост. 3-его числа вариантов $[2; 17] \in Z$
всего 16 вар

$6 \cdot 16 = 96$ вар.

$k_2 \, l_2 \, m_2$

$\underset{3 вар}{16}$ $\underset{2 вар}{1}$ 6 вар.

для 3-его $[2; 15] \in Z$
всего 14 вар.

$6 \cdot 14 = 84$ вар.

для каждой расст. $k_1 l_1 m_1$ есть 84 вар. расстановки $k_2 \, l_2 \, m_2$

Итого $84 \cdot 96 = 8064$

Ответ: 8064

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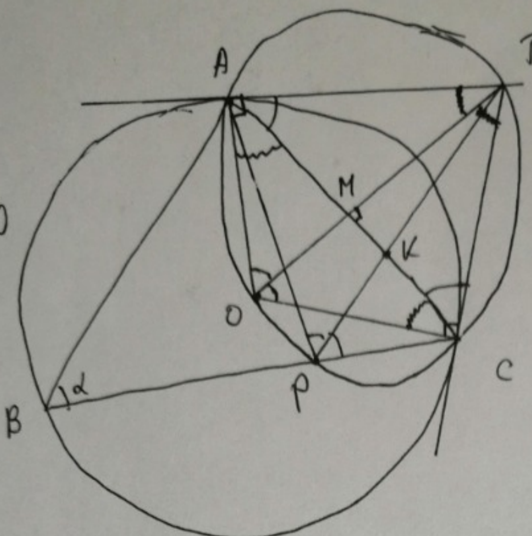
Чистовик

№6)



1) $\dfrac{S_{CPK}}{S_{APK}} = \dfrac{9}{12} = \dfrac{3}{4} = \dfrac{CK\cdot\frac{1}{2}h}{AK\cdot\frac{1}{2}h} = \dfrac{CK}{AK}$

$\dfrac{CK}{AK} = \dfrac{3}{4}$

2) ΔATC р/б т.к. $AT = TC$ (отр. касательных из 1 точки)

$\Rightarrow \angle ATO = \angle CTO = \beta$

По th. о впис. углах опир. на одну дугу

$\angle OAC = \beta = \angle ACO$

аналогично $\angle AOT = \angle COT = \angle APT = \angle CPT = \angle TAC = \angle TCA = \alpha$

$\angle ABC = \angle TAC = \alpha$ (по th. об угле между кас. и хордой пров. в (.) кас.)

$\angle APC$ - внешний к $\Delta APB \Rightarrow 2\alpha = \alpha + \angle BAP \Rightarrow \angle BAP = \alpha$

$\Rightarrow BP = AP$

$\dfrac{AP}{CP} = \dfrac{AK}{CK} = \dfrac{4}{3}$ по th. о бис-се

$\dfrac{BP}{CP} = \dfrac{4}{3}$ $\quad BP = \frac{4}{3}CP;\ \frac{7}{3}CP = BC$

$\Rightarrow \dfrac{S_{CPA}}{S_{ABC}} = \dfrac{\frac{1}{2}h\cdot CP}{\frac{1}{2}h\cdot BC} = \dfrac{CP}{BC} = \dfrac{3}{7}$ $\quad \dfrac{CP}{BC} = \dfrac{3}{7}$

$S_{ABC} = \frac{7}{3}\cdot S_{CPA} = \frac{7}{3}(12+9) = 49$

$\operatorname{arctg}\frac{3}{7} = \alpha$

$\Rightarrow \operatorname{tg}\alpha = \frac{3}{7}$

$\sin\alpha = \frac{3}{a}$

$\cos\alpha = \frac{7}{a}$

$\frac{9+49}{a^2} = 1 \Rightarrow a = \sqrt{58}$

$\sin\alpha = \frac{3}{\sqrt{58}}$

$\cos\alpha = \frac{7}{\sqrt{58}}$ $\quad \cos 2\alpha = 2\cdot\frac{49}{58} - 1 = \frac{40}{58}$

$AP = 4y$

$CP = 3y$

$S_{APK} = \frac{1}{2}PK\cdot 4y\sin\alpha = 12$

$S_{CPK} = \frac{1}{2}PK\,y\cdot 3\sin\alpha = 9$

$S_{APC} = \frac{1}{2}12y^2\cdot\sin 2\alpha = 12$

$PK\cdot y\sin\alpha = 6$ $\quad PK^2\cdot y^2\sin^2\alpha = 36$

$y^2\sin\alpha = \frac{36}{PK^2\sin\alpha}$

$y^2\cdot\sin\alpha = \frac{7}{4\cos\alpha} = \frac{36}{PK^2\sin\alpha}$ $\quad PK = \frac{12}{\sqrt{7\operatorname{tg}\alpha}} = 4\sqrt{3}$

$y = \dfrac{6}{4\sqrt{3}\cdot\sin(\operatorname{arctg}\frac{3}{7})} = \dfrac{\sqrt{58}}{\sqrt{12}}$

По th. косинусов

$AC^2 = \frac{16\cdot 58}{12} + \frac{9\cdot 58}{12} - 2\cdot 3\cdot 4\cdot\frac{58}{12}\cdot\frac{40}{58}$

$AC^2 = \frac{25\cdot 58 - 24\cdot 40}{12} = \frac{490}{12} \Rightarrow AC = \frac{7\sqrt{10}}{2\sqrt{3}}$

Ответ: а) $S_{ABC} = 49$

б) $AC = \frac{7\sqrt{10}}{2\sqrt{3}}$

Черновик

11)

$$\begin{cases} НОД(a;b;c) = 35 \\ НОК(a;b;c) = 5^{18} \cdot 7^{16} \end{cases} \Rightarrow \begin{matrix} a \vdots 35 \\ b \vdots 35 \\ c \vdots 35 \end{matrix}$$

$$] \; a = 5^{k_1} \cdot 7^{k_2}$$
$$b = 5^{l_1} \cdot 7^{l_2}$$
$$c = 5^{m_1} \cdot 7^{m_2}$$

$$\Rightarrow \left[\begin{matrix} k_1 = 18 \\ l_1 = 18 \\ m_1 = 18 \end{matrix} \right. \quad \left[\begin{matrix} k_2 = 16 \\ l_2 = 16 \\ m_2 = 16 \end{matrix} \right.$$

$\Rightarrow$ т.к. НОД $= 35 \Rightarrow$ $\left[\begin{matrix} k_2 = 1 \\ l_2 = 1 \\ m_2 = 1 \end{matrix} \right. \quad \left[\begin{matrix} k_1 = 1 \\ l_1 = 1 \\ m_1 = 1 \end{matrix} \right.$

$k_1 \quad l_1 \quad m_1$

3   2   6 вар.

18   1

третье $\in [2; 17]$ 16 вар. $\quad \underline{6 \cdot 16 = 96}$

$k_2 \quad l_2 \quad m_2$

$\frac{3}{16} \quad \frac{2}{1}$ 6 вар.

Третье $\in [2; 15]$ 14 вар $\quad \underline{6 \cdot 14 = 84}$

для каждой расстановки $k_1 l_1 m_1$ есть 84 расст. $k_2 l_2 m_2$

$\Rightarrow 96 \cdot 84 = \underline{\underline{8064}}$

$$\begin{array}{r} 96 \\ \times\ 84 \\ \hline 384 \\ 768 \\ \hline 8064 \end{array}$$

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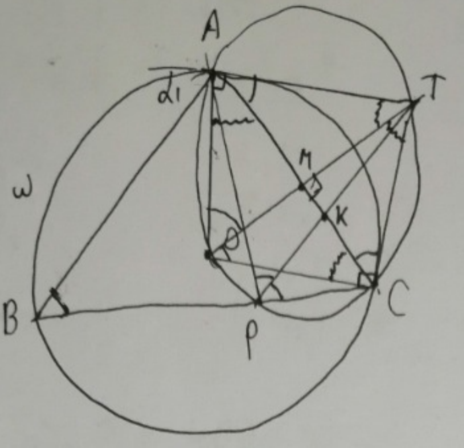
Черновик

№6)



1) $\frac{S_{CPK}}{S_{APK}} = \frac{9}{12} = \frac{3}{4}$

$\Rightarrow \frac{CK}{AK} = \frac{3}{4}$

$\Rightarrow$ OATC - дельтоид

$\Rightarrow AC \perp OT$

($\triangle ACT$ р/б $\Rightarrow TM \perp AC$)

$\frac{AP}{CP} = \frac{AK}{CK} = \frac{4}{3}$ $AP = 4y$, $CP = 3y$

$S_{APK} = \frac{1}{2} PK \cdot 4y \cdot \sin\alpha = 12$

$PK \cdot y \cdot \sin\alpha = 6$

$S_{CPK} = \frac{1}{2} PK \cdot 3y \sin\alpha = 9$

$PK\, y \sin\alpha = 6$

$\frac{1}{2} 4y \cdot 3y \cdot \sin 2\alpha = 21$

$6y^2 \sin 2\alpha = 21$

$y^2 \sin 2\alpha = \frac{7}{2}$

$\sin 2\alpha = \frac{7}{2y^2}$

$PK^2 y^2 \sin^2\alpha = 36$

$y^2 \sin\alpha = \frac{36}{PK^2 \sin\alpha}$

$y^2 \cdot \sin\alpha = \frac{7}{4\cos\alpha} = \frac{36}{PK^2 \sin\alpha}$

$58 \times 25 = 290 + 1160 = 1450$

$24 \times 40 = 960$

$1450 - 960 = 490$

$) = \alpha$

$\angle ACB = \angle\alpha$,

$\angle TAC = \angle ABC$

$\cos\alpha = \frac{7}{\sqrt{58}}$

$\angle APC$ - внешн. к $\triangle APB \Rightarrow \angle BAP = \alpha$

$\Rightarrow \underline{BP = AP}$

$\cos 2\alpha = 2 \cdot \frac{49}{58} - 1 = \frac{98-58}{58} = \frac{40}{58}$

$AC^2 = \frac{16 \cdot 58}{12} + \frac{9 \cdot 58}{12} - 2 \cdot 3 \cdot 4 \cdot \frac{58}{12} \cdot \frac{40}{58}$

$AC^2 = \frac{25 \cdot 58}{12} - \frac{24 \cdot 40}{12} = \frac{490}{12}$

$\frac{49 \cdot 10}{12} = \frac{49 \cdot 5}{6}$

$\frac{36 \cdot 4}{7\, tg\alpha} = PK^2$

$\frac{2 \cdot 6}{\sqrt{7 tg\alpha}} = PK = 4\sqrt{3}$

$y = \frac{8}{4\sqrt{3} \cdot \sin(arctg \frac{3}{7})} = \frac{\sqrt{3}}{2 \cdot \frac{3}{\sqrt{58}}} = \frac{\sqrt{58}}{\sqrt{12}}$

Черновик

№2) $\log_{\sqrt{2x-3}}(x+1) \cdot \quad \log_{2x^2-3x+5}(2x-3)^2 \quad \log_{x+1} 2x^2-3x+5$

(1) (2) (3)

I) (1)=(2) $\log_{\sqrt{2x-3}}(x+1) = \log_{2x^2-3x+5}(2x-3)^2$

$\log_{2x-3}(x+1) = \log_{2x^2-3x+5}(2x-3)$

$\log_{2x-3}(x+1) \cdot \log_{2x-3}(2x^2-3x+5) = \cancel{0}\ 1$

$\log_{x+1} 2x^2-3x+5 + 1 = \log_{2x^2-3x+5}(2x-3)^2$

$\log_{x+1}((2x^2-3x+5)(x+1)) = \log_{2x^2-3x+5}(2x-3)^2$

II) (1)=(3) $\log_{\sqrt{2x-3}}(x+1) = \log_{x+1} 2x^2-3x+5$

$\log_{2x^2-3x+5}(2x-3)^2 = \log_{x+1}\frac{2x^2-3x+5}{x+1}$

$4x^2-6x+9$

] $\cancel{(2x-3)^2 = u}$

$2x^2-3x+5 = u \geq \frac{1}{2}$

$\log_u(2u-1) = \log_{x+1}\frac{u}{x+1}$

$\log_u(2u^2-u) \cdot \log_u x+1 = 1$

$u_0 = \frac{1}{4}$